On finite groups with some primary subgroups satisfying partial S-Π-property*

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Abstract

A p-subgroup H of a finite group G is said to satisfy partial S-Π-property in G if G has a chief series $\Gamma_G : 1 = G_0 < G_1 < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((H \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number. In this paper, we mainly investigate the structure of finite groups with some primary subgroups satisfying partial S-Π-property.

1 Introduction

Throughout this paper, all groups considered are finite. G always denotes a group, p denotes a prime, and $|G|_p$ denotes the order of Sylow p-subgroups of G. Also, we use $\mathfrak{U}$ and $\mathfrak{N}$ to denote the classes of all supersoluble groups and nilpotent groups, respectively.

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Recall that a subgroup H of G has the cover-avoidance property in G or H is called a CAP-subgroup of G if either H covers L/K (i.e. $L \leq HK$) or H avoids L/K (i.e. $H \cap L \leq K$) for each G-chief factor L/K. Also, a subgroup H of G is said to be S-quasinormally embedded [5] in G if each Sylow subgroup of H is also a Sylow subgroup of some S-quasinormal subgroup of G. The CAP-subgroups and S-quasinormally embedded subgroups play an important role in the study of the structure of finite groups, and have been investigated by many authors. As a generalization of CAP-subgroup and S-quasinormally embedded subgroup, W. Guo, A. N. Skiba and N. Yang introduced the concept of generalized CAP-subgroup [14]: a subgroup H of G is said to be a generalized CAP-subgroup of G if for each G-chief factor L/K, either H avoids L/K or the following hold: (1) If L/K is non-abelian, then $|L : (H \cap L)K|$ is a p'-number for every $p \in \pi((H \cap L)K/K)$; (2) If L/K is a p-group, then $|G : N_G((H \cap L)K)|$ is a p-number. The authors in [14] showed that every CAP-subgroup and every S-quasinormally embedded subgroup of G are both a generalized CAP-subgroup of G, and the converse is not true. In connection with this, A. N. Skiba proposed the following question in Seminar of USTC, 2014:

Question 1.1. (see also [12, Chap. 1, Problem 6.14]). *To study the structure of finite groups when the condition of every chief factor in the generalized CAP-subgroup is replaced by every chief factor in some chief series.*

The main objective of the paper is to give an answer to Question 1.1. For this purpose, we now introduce the following concept:

Definition 1.2. *A p-subgroup H of G is said to satisfy partial S-Π-property in G if G has a chief series $\Gamma_G : 1 = G_0 < G_1 < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((H \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number.*

It is clear that a p-subgroup H of G satisfy partial S-Π-property in G if H is a generalized CAP-subgroup of G. But the next example illustrates that the converse is not true.

Example 1.3. Let $L_1 = \langle a, b \mid a^5 = b^5 = 1, ab = ba\rangle$ and $L_2 = \langle a', b'\rangle$ be a copy of L_1. Let α be an automorphism of L_1 of order 3 satisfying that $a^\alpha = b$, $b^\alpha = a^{-1}b^{-1}$. Put $G = (L_1 \times L_2) \rtimes \langle \alpha\rangle$ and $H = \langle a\rangle \times \langle a'\rangle$. Then G has a minimal normal subgroup N such that $H \cap N = 1$. Note that $\mathit{\Gamma}_G : 1 < N < HN < G$ is a chief series of G. Then H satisfies partial S-Π-property in G. But since $|G : N_G(H \cap L_1)| = |G : N_G(\langle a\rangle)| = 3$, H is not a generalized CAP-subgroup of G.

Note also that, X. Chen and W. Guo in [6] introduced the concept of partial Π-property: a subgroup H of G satisfies partial Π-property in G if there exists a chief series $\Gamma_G : 1 =$

$G_0 < G_1 < \cdots < G_n = G$ of G such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, $|G/G_{i-1} : N_{G/G_{i-1}}((H \cap G_i)G_{i-1}/G_{i-1})|$ is a $\pi((H \cap G_i)G_{i-1}/G_{i-1})$-number. It is easy to see that if a p-subgroup H of G satisfies partial Π-property in G, then H satisfies partial S-Π-property in G. However, the converse does not hold in general.

Example 1.4. Let $G = A_5$ and H be a Sylow 5-subgroup of A_5, where A_5 is an alternative group of degree 5. Then it is easy to see that H satisfies partial S-Π-property in G. However, since $|G : N_G(H)|$ is not a 5-number, we have that H does not satisfy partial Π-property in G.

Let $\mathfrak{F}$ be a formation. The $\mathfrak{F}$-residual of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with quotient in $\mathfrak{F}$. A G-chief factor L/K is said to be $\mathfrak{F}$-central in G if $L/K \rtimes G/C_G(L/K) \in \mathfrak{F}$. A normal subgroup N of G is called $\mathfrak{F}$-hypercentral in G if either $N = 1$ or every G-chief factor below N is $\mathfrak{F}$-central in G. Let $Z_{\mathfrak{F}}(G)$ denote the $\mathfrak{F}$-hypercentre of G, that is, the product of all $\mathfrak{F}$-hypercentral normal subgroups of G. Moreover, the generalized Fitting subgroup $F^*(G)$ (resp. the generalized p-Fitting subgroup $F_p^*(G)$) of G is quasinilpotent radical (resp. p-quasinilpotent radical) of G (for details, see [18, Chap. X] and [4]). We denote the Fitting subgroup and the p-Fitting subgroup of G by $F(G)$ and $F_p(G)$, respectively.

In this paper, we arrive at the following main results.

Theorem 1.5. *Let E and X be normal subgroups of G such that $F^*(E) \leq X \leq E$. Suppose that for any non-cyclic Sylow subgroup P of X, every maximal subgroup of P satisfies partial S-Π-property in G, or every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) satisfies partial S-Π-property in G. Then $E \leq Z_{\mathfrak{U}}(G)$.*

Theorem 1.6. *Let E and X be p-soluble normal subgroups of G such that $F_p(E) \leq X \leq E$. Suppose that X has a Sylow p-subgroup P such that every maximal subgroup of P satisfies partial S-Π-property in G, or every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) satisfies partial S-Π-property in G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$.*

All unexplained notation and terminology are standard, as in [3, 8, 11].

2 Preliminaries

Firstly, we present some basic properties of partial S-Π-property as follows.

Lemma 2.1. *Suppose that a p-subgroup H of G satisfies partial S-Π-property in G and $N \trianglelefteq G$.*

(1) *If $H \leq N$, then H satisfies partial S-Π-property in N.*

(2) *If either* $N \leq H$ *or* $(p, |N|) = 1$, *then* HN/N *satisfies partial* S-Π-*property in* G/N.

(3) *If every maximal subgroup of a Sylow* p-*subgroup* P *of* G *satisfies partial* S-Π-*property in* G, *then every maximal subgroup of* PN/N *also satisfies partial* S-Π-*property in* G/N.

Proof. By the hypothesis, we may assume that G has a chief series $\Gamma_G : 1 = G_0 < G_1 < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((H \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number.

(1) Obviously, $\Gamma_N : 1 = G_0 \cap N \leq G_1 \cap N \leq \cdots \leq G_n \cap N = N$ is a normal series of N. Let L/K be an N-chief factor such that $G_{i-1} \cap N \leq K \leq L \leq G_i \cap N$ $(1 \leqslant i \leqslant n)$. If $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1}, then $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of $(G_i \cap N)G_{i-1}/G_{i-1}$. We can deduce that $(H \cap G_i)(G_{i-1} \cap N)/(G_{i-1} \cap N)$ is a Sylow p-subgroup of $(G_i \cap N)/(G_{i-1} \cap N)$. Hence $(H \cap L)K/K$ is a Sylow p-subgroup of L/K. Now assume that $|G : N_G((H \cap G_i)G_{i-1})|$ is a p-number. Then $|N : N_N((H \cap G_i)(G_{i-1} \cap N))|$ is a p-number because $N_G((H \cap G_i)G_{i-1}) \leq N_G((H \cap G_i)G_{i-1} \cap N)$. It is easy to see that $N_N((H \cap G_i)(G_{i-1} \cap N)) \leq N_N((H \cap L)K)$, and so $|N : N_N((H \cap L)K)|$ is a p-number. Hence H satisfies partial S-Π-property in N.

(2) Note that if either $N \leq H$ or $(p, |N|) = 1$, then $HN \cap XN = (H \cap X)N$ for every normal subgroup X of G. Now consider the normal series $\Gamma_{G/N} : 1 = G_0N/N \leq G_1N/N \leq \cdots \leq G_nN/N = G/N$ of G/N. For every normal section $G_iN/G_{i-1}N$, we have that $(HN \cap G_iN)G_{i-1}N = HG_{i-1}N \cap G_iN = (H \cap G_i)G_{i-1}N$. If $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1}, then $(HN \cap G_iN)G_{i-1}N/G_{i-1}N = (H \cap G_i)G_{i-1}N/G_{i-1}N$ is a Sylow p-subgroup of $G_iN/G_{i-1}N$. Now assume that $|G : N_G((H \cap G_i)G_{i-1})|$ is a p-number. Then $|G : N_G((HN \cap G_iN)G_{i-1}N)| = |G : N_G((H \cap G_i)G_{i-1}N)|$ is a p-number. Therefore, HN/N satisfies partial S-Π-property in G/N.

(3) Let T/N be any maximal subgroup of PN/N. Then there exists a maximal subgroup P_1 of P such that $T = P_1N$ and $P_1 \cap N = P \cap N$. It is easy to derive that $P_1N \cap XN = (P_1 \cap X)N$ for any normal subgroup X of G. With a similar argument as (2), we have that T/N satisfies partial S-Π-property in G/N. $\square$

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.2. [7, Lemma 2.12] *Let* P *be a normal* p-*subgroup of* G *and* C *a Thompson critical subgroup of* P (see [10, *p.* 186]). *If* $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$ *or* $C \leq Z_{\mathfrak{U}}(G)$ *or* $\Omega(P) \leq Z_{\mathfrak{U}}(G)$, *then* $P \leq Z_{\mathfrak{U}}(G)$.

The next lemma is evident.

Lemma 2.3. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$ and N a normal subgroup of G such that $|N|_p \leq p$. If G/N is p-nilpotent, then G is p-nilpotent.*

Lemma 2.4. [7, Lemma 2.11] *Let P be a p-group of nilpotent class at most 2. Suppose that the exponent of $P/Z(P)$ divides p.*

(1) *If $p > 2$, then the exponent of $\Omega(P)$ is p.*

(2) *If P is a non-abelian 2-group, then the exponent of $\Omega(P)$ is 4.*

Lemma 2.5. [26, Theorem B] *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

Lemma 2.6. [2, Theorem 2.1.6] *Let G be a p-supersoluble group. Then the derived subgroup G' of G is p-nilpotent. In particular, if $O_{p'}(G) = 1$, then G has a unique Sylow p-subgroup.*

3 Proof of Main Results

The following propositions are the main steps of the proof of Theorems 1.5 and 1.6.

Proposition 3.1. *Let P be a normal p-subgroup of G. If every maximal subgroup of P satisfies partial S-Π-property in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Suppose that this proposition is false, and let (G, P) be a counterexample for which $|G| + |P|$ is minimal. Then:

(1) *There is a unique minimal normal subgroup N of G contained in P, $P/N \leq Z_{\mathfrak{U}}(G/N)$ and $|N| > p$.*

Let N be any minimal normal subgroup of G contained in P. Then clearly, $(G/N, P/N)$ satisfies the hypothesis by Lemma 2.1(2), and so the choice of (G, P) yields that $P/N \leq Z_{\mathfrak{U}}(G/N)$. If $|N| = p$, then $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $|N| > p$. Now suppose that G has a minimal normal subgroup R contained in P such that $N \neq R$. With a similar discussion as above, we obtain that $P/R \leq Z_{\mathfrak{U}}(G/R)$. It follows that $NR/R \leq Z_{\mathfrak{U}}(G/R)$, and so $|N| = p$, a contradiction.

(2) $\Phi(P) = 1$*, and so P is elementary abelian.*

If $\Phi(P) \neq 1$, then by (1), $N \leq \Phi(P)$. This induces that $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$ because $P/N \leq Z_{\mathfrak{U}}(G/N)$, and so $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.2. This contradiction shows that $\Phi(P) = 1$, and so P is elementary abelian.

(3) *The final contradiction.*

Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup of G, say G_p. Then $P_1 = N_1S$ is a maximal subgroup of P, where S is a complement of N in P.

By the hypothesis, G has a chief series $\Gamma_G : 1 = G_0 < G_1 < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(P_1 \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((P_1 \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number. Obviously, there exists an integer k $(1 \leqslant k \leqslant n)$ such that $G_k = G_{k-1} \times N$. It follows from (1) that $P \cap G_{k-1} = 1$. If $(P_1 \cap G_k)G_{k-1}/G_{k-1}$ is a Sylow p-subgroup of G_k/G_{k-1}, then $G_k = (P_1 \cap G_k)G_{k-1}$, and thus $N \leq G_k \leq P_1G_{k-1}$. This implies that $N \leq P_1G_{k-1} \cap P \leq P_1$, a contradiction. Now assume that $|G : N_G((P_1 \cap G_k)G_{k-1})|$ is a p-number. Then $|G : N_G(P_1G_{k-1} \cap N))|$ is a p-number. Since $N_1 \leq P_1G_{k-1} \cap N < N$, we have that $N_1 = P_1G_{k-1} \cap N$. Hence $N_1 \trianglelefteq G$ because $N_1 \trianglelefteq G_p$, and so $N_1 = 1$, which is also a contradiction. The proof is thus completed. □

Proposition 3.2. *Let E be a normal subgroup of G and p a prime divisor of $|E|$ with $(|E|, p-1) = 1$. Suppose that E has a Sylow p-subgroup P such that every maximal subgroup of P satisfies partial S-Π-property in G. Then E is p-nilpotent.*

Proof. Suppose that this proposition is false, and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Now we proceed the proof via the following steps.

(1) $O_{p'}(E) = 1$.

If $O_{p'}(E) \neq 1$, then by Lemma 2.1(2), $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis. The choice of (G, E) yields that $E/O_{p'}(E)$ is p-nilpotent, and so E is p-nilpotent, a contradiction. Hence $O_{p'}(E) = 1$.

(2) $E = G$.

Assume that $E < G$. Then by Lemma 2.1(1), the hypothesis holds for (E, E). By the choice of the (G, E), E is p-nilpotent. This contradiction shows that $E = G$.

(3) *G has a unique minimal normal subgroup N, G/N is p-nilpotent and $\Phi(G) = 1$.*

Let N be any minimal normal subgroup of G. Then by Lemma 2.1(3), the hypothesis still holds for $(G/N, G/N)$. The choice of (G, E) yields that G/N is p-nilpotent. Hence N is the unique minimal normal subgroup of G, and it is easy to see that $\Phi(G) = 1$.

(4) *The final contradiction.*

If $P \cap N \leq \Phi(P)$, then by [17, Chap. IV, Satz 4.7], N is p-nilpotent. Hence by (1), N is a p-group, and so $N \leq \Phi(G) = 1$, which is impossible. Thus $P \cap N \nleq \Phi(P)$. Then P has a maximal subgroup P_1 such that $P = P_1(P \cap N)$. By the hypothesis and (3), G has a chief series $\Gamma_G : 1 = G_0 < G_1 = N < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(P_1 \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((P_1 \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number. Evidently, $P_1 \cap N$ is not a Sylow p-subgroup of N. Therefore, $|G : N_G(P_1 \cap N)|$ is a p-number. Since $P \leq N_G(P_1 \cap N)$, we have that $P_1 \cap N \trianglelefteq G$. It follow that $P_1 \cap N = 1$, and thereby $|N|_p = p$. Then by (3) and Lemma 2.3, G is p-nilpotent, a contradiction. This completes the proof. □

Proposition 3.3. *Let P be a normal p-subgroup of G. If every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) satisfies partial S-Π-property in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Suppose that this proposition is false, and let (G, P) be a counterexample for which $|G| + |P|$ is minimal. Then:

(1) *G has a unique normal subgroup N such that P/N is a G-chief factor, $N \leq Z_{\mathfrak{U}}(G)$ and $|P/N| > p$.*

Let P/N be a G-chief factor. Then (G, N) satisfies the hypothesis, and the choice of (G, P) implies that $N \leq Z_{\mathfrak{U}}(G)$. If $|P/N| = p$, then $P/N \leq Z_{\mathfrak{U}}(G/N)$, and so $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Thus $|P/N| > p$. Now assume that P/R is a G-chief factor with $N \neq R$. With a similar argument as above, we have that $R \leq Z_{\mathfrak{U}}(G)$. This yields that $P = NR \leq Z_{\mathfrak{U}}(G)$, a contradiction occurs. Therefore, N is the unique normal subgroup of G such that P/N is a G-chief factor.

(2) *The exponent of P is p or 4 (when P is a non-abelian 2-group).*

Let D be a Thompson critical subgroup of P. Then the nilpotent class of D is at most 2 and $D/Z(D)$ is elementary abelian by [10, Chap. 5, Theorem 3.11]. If $\Omega(D) < P$, then $\Omega(D) \leq N \leq Z_{\mathfrak{U}}(G)$ by (1). It follows from Lemma 2.2 that $P \leq Z_{\mathfrak{U}}(G)$, against supposition. Thus $P = D = \Omega(D)$. Then by Lemma 2.4, the exponent of P is p or 4 (when P is a non-abelian 2-group).

(3) *The final contradiction.*

Let G_p be a Sylow p-subgroup of G. Since $P/N \cap Z(G_p/N) > 1$, there exists a subgroup T/N of $P/N \cap Z(G_p/N)$ of order p. Let $x \in T \backslash N$ and $H = \langle x \rangle$. Then $T = HN$ and $|H| = p$ or 4 (when P is a non-abelian 2-group) by (2). By the hypothesis, G has a chief series $\Gamma_G : 1 = G_0 < G_1 < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(H \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((H \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number. Clearly, there exists an integer k $(1 \leqslant k \leqslant n)$ such that $P \leq G_k$ and $P \nleq G_{k-1}$. Since N is the unique normal subgroup of G such that P/N is a G-chief factor by (1), we have that $P \cap G_{k-1} \leq N$. If $G_k = NG_{k-1}$, then $P = N(P \cap G_{k-1}) = N$. This contradiction forces that $N \leq G_{k-1}$, and so $P \cap G_{k-1} = N$.

Firstly suppose that HG_{k-1}/G_{k-1} is a Sylow p-subgroup of G_k/G_{k-1}. Then $P \leq HG_{k-1}$. This implies that $P = H(P \cap G_{k-1}) = HN = T$, and so $|P/N| = |T/N| = p$, which contradicts (1). Now assume that $|G : N_G(HG_{k-1})|$ is a p-number, and so $|G : N_G(T)|$ is a p-number. Since $G_p \leq N_G(T)$, we have that $T \trianglelefteq G$. It follows from (1) that $P = T$ because $H \neq N$, a contradiction also occurs. This ends the proof. $\square$

Proposition 3.4. *Let E be a normal subgroup of G and p a prime divisor of $|E|$ with $(|E|, p-1) = 1$. Suppose that E has a Sylow p-subgroup P such that every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) satisfies partial S-Π-property in G. Then E is p-nilpotent.*

Proof. Suppose that this proposition is false, and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Now we proceed the proof via the following steps.

(1) *$O_{p'}(G) = 1$ and $E = G$.*

With a similar argument as in steps (1) and (2) of the proof of Proposition 3.2, we have that $O_{p'}(G) = 1$ and $E = G$.

(2) *$Z(G)$ is the unique normal subgroup of G such that $G/Z(G)$ is a G-chief factor, $Z(G) = Z_\infty(G) = O_p(G)$ and $G^{\mathfrak{N}} = G$.*

Let G/K be a G-chief factor. Then by Lemma 2.1(1), (K, K) satisfies the hypothesis. The choice of (G, E) yields that K is p-nilpotent, and so $K \le F_p(G)$. Since G is not p-nilpotent, $K = F_p(G)$. This shows that $F_p(G)$ is the unique normal subgroup of G such that $G/F_p(G)$ is a G-chief factor. By (1) and Proposition 3.3, $F_p(G) = O_p(G) \le Z_{\mathfrak{U}}(G)$. As $(|G|, p-1) = 1$, we have that $O_p(G) \le Z_\infty(G)$, and thereby $Z_\infty(G) = O_p(G)$ because $Z_\infty(G) \le F_p(G) = O_p(G)$. If $G^{\mathfrak{N}} < G$, then $G^{\mathfrak{N}} \le Z_\infty(G)$. This implies that G is p-nilpotent, a contradiction. Thus $G^{\mathfrak{N}} = G$. Then by [8, Chap. IV, Theorem 6.10], $Z_\infty(G) \le Z(G)$, and so $Z_\infty(G) = Z(G)$.

(3) *P is non-abelian.*

If P is abelian, then by (1) and [17, Chap. VI, Satz 14.3], $G' \cap Z(G) = O_{p'}(G) = 1$. Since $G' = G$ by (2), we have that $Z(G) = 1$, and so G is a simple group. Let x be an element of G of order p. Then by the hypothesis, either $\langle x \rangle$ is a Sylow p-subgroup of G or $|G : N_G(\langle x \rangle)|$ is a p-number. In the former case, G is p-nilpotent by Lemma 2.3, a contradiction. In the latter case, $\langle x \rangle \le O_p(G) = 1$ by (2), also a contradiction. Thus P is non-abelian.

(4) *The final contradiction.*

Suppose that all cyclic subgroups of P of order p and 4 are contained in $Z(G)$, then G is p-nilpotent by [17, Chap. IV, Satz 5.5]. Hence G has an element x of order p or 4 such that $x \notin Z(G)$. Then by (2), (3) and the hypothesis, G has a chief series $\Gamma_G : 1 = G_0 < G_1 < \cdots < G_{n-1} = Z(G) < G_n = G$ such that for every G-chief factor G_i/G_{i-1} $(1 \leqslant i \leqslant n)$ of Γ_G, either $(\langle x \rangle \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((\langle x \rangle \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number. If $\langle x \rangle Z(G)/Z(G)$ is a Sylow p-subgroup of $G/Z(G)$, then $P = \langle x \rangle Z(G)$, and so P is abelian, which contradicts (3). Now assume that $|G : N_G(\langle x \rangle Z(G))|$ is a p-number. Then $\langle x \rangle Z(G) \le O_p(G) = Z(G)$ by (2). This contradiction completes the proof. $\square$

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let p be the smallest prime divisor of $|X|$ and X_p a Sylow p-subgroup of X. If X_p is cyclic, then X is p-nilpotent by [24, 10.1.9]. Now assume that X_p is not cyclic. Then by Lemma 2.1(1), Propositions 3.2 and 3.4, X is also p-nilpotent. Let $X_{p'}$ be the normal p-complement of X. Then $X_{p'} \trianglelefteq G$. If X_p is cyclic, then clearly, $X/X_{p'} \leq Z_{\mathfrak{U}}(G/X_{p'})$. Now assume that X_p is not cyclic. Then it is easy to see that $(G/X_{p'}, X/X_{p'})$ satisfies the hypothesis of Proposition 3.1 or Proposition 3.3 by Lemma 2.1(2). Hence we also have that $X/X_{p'} \leq Z_{\mathfrak{U}}(G/X_{p'})$.

Let q be the second smallest prime divisor of $|X|$. By arguing similarly as above, we obtain that $X_{p'}$ is q-nilpotent and $X_{p'}/X_{\{p,q\}'} \leq Z_{\mathfrak{U}}(G/X_{\{p,q\}'})$, where $X_{\{p,q\}'}$ is the normal q-complement of $X_{p'}$. The rest can be deduced by analogy. Hence we can obtain that $X \leq Z_{\mathfrak{U}}(G)$. Then by Lemma 2.5, $E \leq Z_{\mathfrak{U}}(G)$. The theorem is thus proved. $\square$

In order to prove Theorem 1.6, we need the following proposition.

Proposition 3.5. *Let E be a p-soluble normal subgroup of G. Suppose that E has a Sylow p-subgroup P such that every maximal subgroup of P satisfies partial S-Π-property in G, or every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) satisfies partial S-Π-property in G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$.*

Proof. Suppose that this proposition is false, and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. With a similar discussion as in step (1) of the proof of Proposition 3.2, we have that $O_{p'}(E) = 1$. If E is p-supersoluble, then $E' \leq F(E) = P$ by Lemma 2.6. Hence E is soluble, and so $F^*(E) = F(E) = P$ by [18, Chap. X, Corollary 13.7(d)]. Note that by Propositions 3.1 and 3.3, we have that $P \leq Z_{\mathfrak{U}}(G)$. It follows from Lemma 2.5 that $E \leq Z_{\mathfrak{U}}(G)$, which is impossible. Thus E is not p-supersoluble. If $E < G$, then since (E, E) satisfies the hypothesis by Lemma 2.1(1), E is p-supersoluble by the choice of (G, E). This contradiction implies that $E = G$ and G is not p-supersoluble.

Firstly suppose that every maximal subgroup of P satisfies partial S-Π-property in G. Let N be a minimal normal subgroup of G. Since G is p-soluble and $O_{p'}(G) = 1$, we see that $N \leq O_p(G)$. By Lemma 2.1(2), the hypothesis holds for $(G/N, G/N)$, and so the choice of (G, E) implies that G/N is p-supersoluble. Then it is easy to see that N is the unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Hence P has a maximal subgroup P_1 such that $P = P_1N$. Then by the hypothesis, G has a chief series $\Gamma_G : 1 = G_0 < G_1 = N < \cdots < G_n = G$ such that for every G-chief factor G_i/G_{i-1} ($1 \leqslant i \leqslant n$) of Γ_G, either $(P_1 \cap G_i)G_{i-1}/G_{i-1}$ is a Sylow p-subgroup of G_i/G_{i-1} or $|G/G_{i-1} : N_{G/G_{i-1}}((P_1 \cap G_i)G_{i-1}/G_{i-1})|$ is a p-number. Since $N \nleq P_1$, we have that $|G : N_G(P_1 \cap N)|$ is a p-number. Hence $P_1 \cap N \trianglelefteq G$. It follows that $P_1 \cap N = 1$, and so $|N| = p$. Thus G is p-supersoluble, a contradiction.

Now assume that every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) satisfies partial S-Π-property in G. Let G/K be a G-chief factor. Then G/K is p-supersoluble because G/K is a p-soluble simple group, and (K, K) satisfies the hypothesis by Lemma 2.1(1). By the choice of (G, E), K is p-supersoluble. Since $O_{p'}(K) \leq O_{p'}(G) = 1$, $P \cap K \trianglelefteq G$ by Lemma 2.6. Then by Proposition 3.3, $P \cap K \leq Z_{\mathfrak{U}}(G)$. As $G/(P \cap K)$ is p-supersoluble, we have that G is p-supersoluble. The final contradiction completes the proof. □

Proof of Theorem 1.6. Note that $O_{p'}(X) = O_{p'}(E)$. Then by Proposition 3.5, we have that $X/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$, and so $F_p(E)/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$. It follows from [4, Lemma 2.10] that $F^*(E/O_{p'}(E)) = F_p^*(E/O_{p'}(E)) = F_p(E)/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$ because E is p-soluble. Hence $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$ by Lemma 2.5. □

4 Final Remarks

In this section, we shall show that the concept of partial S-Π-property can be viewed as a generalization of many known embedding properties. Though some of them are generalized by the concept of partial Π-property, there are still some embedding properties can only be generalized by the concept of partial S-Π-property as the following proposition illustrates. Hence, as a consequence, a large number of results in former literature can follow directly from our main results.

Proposition 4.1. *Let H be a p-subgroup of G. Then H satisfies partial S-Π-property in G if one of the following holds:*

(1) *H is a generalized CAP-subgroup of G.*

(2) *H satisfies partial Π-property in G.*

(3) *H is Π-normal* [19] *in G.*

(4) *H is $\mathfrak{U}_c$-normal* [1] *in G.*

(5) *H is weakly S-permutable* [25] *in G.*

(6) *H is weakly S-semipermutable* [21] *in G.*

(7) *H is weakly SS-permutable* [15] *in G.*

(8) *H is weakly τ-quasinormal* [22] *in G.*

(9) *H is SE-quasinormal* [7] *in G.*

(10) *H is a partial CAP-subgroup (or semi CAP-subgroup)* [9] *of G.*

(11) *H is S-embedded* [13] *in G.*

(12) *H is $\mathfrak{U}$-quasinormal* [23] *in G.*

(13) *H is $\mathfrak{U}_s$-quasinormal* [16] *in G.*

(14) *H is weakly S-embedded* [20] *in G.*

Proof. Statements (1) and (2) hold by the definition, and statements (3)-(8) and (10)-(13) follow from [6, Lemmas 7.2 and 7.3].

(9) By the definition, G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq H_{seG}$, where H_{seG} denotes the subgroup generated by all subgroups of H which are S-quasinormally embedded in G. Then clearly, $O^p(G) \leq T$. Let $H_1, H_2, \cdots, H_n$ be all subgroups of H which are S-quasinormally embedded in G. Then there exist S-quasinormal subgroups $X_1, X_2, \cdots, X_n$ of G with H_i is a Sylow p-subgroup of X_i $(1 \leq i \leq n)$. If $(X_i)_G = 1$ for all $1 \leq i \leq n$, then by [2, Theorems 1.2.14 and 1.2.17], H_i is S-quasinormal in G for all $1 \leq i \leq n$, and so $H_{seG} = \langle H_1, H_2, \cdots, H_n \rangle$ is S-quasinormal in G. Thus H is weakly S-permutable in G. This implies that H satisfies partial S-Π-property in G by [6, Lemma 7.3(3)]. Hence, without loss of generality, we may assume that $(X_1)_G \neq 1$.

Suppose that $O^p(G) \cap (X_1)_G \neq 1$. Let N be a minimal normal subgroup of G contained in $O^p(G) \cap (X_1)_G$. It is easy to see that HN/N is SE-quasinormal in G/N by [7, Lemma 2.6(3)]. By induction, we have that HN/N satisfies partial S-Π-property in G/N. Since $N \leq X_1$ and H_1 is a Sylow p-subgroup of X_1, $H_1 \cap N$ is a Sylow p-subgroup of N, and so $H \cap N$ is a Sylow p-subgroup of N. This shows that H satisfies partial S-Π-property in G. Now assume that $O^p(G) \cap (X_1)_G = 1$. Let R be a minimal normal subgroup of G contained in $(X_1)_G$. Then $R \leq O_p(G)$. This implies that $R \leq H_1 \leq H$. Since H/R is SE-quasinormal in G/R by [7, Lemma 2.7(2)], we have that H/R satisfies partial S-Π-property in G/R by induction. Therefore, H also satisfies partial S-Π-property in G.

(14) By the definition, G has a normal subgroup T such that HT is S-quasinormal in G and $H \cap T \leq H_{seG}$, where H_{seG} denotes the subgroup generated by all subgroups of H which are S-quasinormally embedded in G. Let $H_1, H_2, \cdots, H_n$ be all subgroups of H which are S-quasinormally embedded in G. Then there exist S-quasinormal subgroups $X_1, X_2, \cdots, X_n$ of G with H_i is a Sylow p-subgroup of X_i $(1 \leq i \leq n)$. Without loss of generality, we assume that $X_i \leq HT$ $(1 \leq i \leq n)$. If $(X_i)_G = 1$ for all $1 \leq i \leq n$, then H_{seG} is S-quasinormal in G. Thus H is S-embedded in G. This yields that H satisfies partial S-Π-property in G by [6, Lemma 7.2(2)]. Hence we may assume that $(X_1)_G \neq 1$.

Suppose that $T \cap (X_1)_G \neq 1$. Then we can obtain that H satisfies partial S-Π-property in G by arguing similarly as in the proof of (9). Now assume that $T \cap (X_1)_G = 1$. Let R be a minimal normal subgroup of G contained in $(X_1)_G$. Since $R \cap T = 1$ and $R \leq HT$, we have that $R \leq H$. With a similar discussion as in the proof of (9), H also satisfies partial S-Π-property in G. □

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